# A New Gauge Fixing Method for Abelian Projection

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February 1, 2008

**Abstract**

We formulate a stochastic gauge fixing method to study the gauge dependence of the Abelian projection. We consider a gauge which interpolates between the maximal Abelian gauge and no gauge fixing. We have found that Abelian dominance for the heavy quark potential holds even in a gauge which is far from maximally Abelian one. The heavy quark potentials from monopole and photon contribution are calculated at several values of the gauge parameter, and the former part shows always the confinement behavior.

# 1 Introduction

Since 'tHooft and Mandelstam proposed the QCD vacuum state to behave like a magnetic superconductor, a dual Meissner effect has been considered to play an essential role in the mechanism of color confinement [1, 2]. A gauge is chosen to reduce the

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gauge symmetry of a non-Abelian group to its maximal Abelian (MA) subgroup, and Abelian fields and magnetic monopole can be identified there. When one reduces $SU(N)$ to $U(1)^{N-1}$ by the partial gauge fixing, monopoles appear in $U(1)^{N-1}$ sector as a topological object. Confinement of QCD is conjectured to be due to condensation of the monopoles. By using the MA gauge which maximizes the functional

$$R = \sum_{x,\mu} \mathrm{Tr}\ U_\mu(x)\sigma_3 U_\mu(x)^\dagger \sigma_3, \tag{1}$$

Suzuki and Yotsuyanagi[3] first found that the value of Abelian string tension is close to that of the non-Abelian theory, where $U_\mu(s)$ are $SU(2)$ link variables on the lattice. Since then many numerical evidences have been collected to show the importance of monopoles in QCD vacuum: we refer to Ref.[4] for a review of these results.

There are infinite ways of extracting $U(1)^{N-1}$ from $SU(N)$. This corresponds to the choice of gauge in Abelian projection. Abelian and monopole dominances can be clearly seen in MA gauge but not in the others; they seem to depend on the choice of gauge in the Abelian projection. However, the dual Meissner effect only in MA gauge is not enough for the proof of color confinement, since Abelian charge confinement and color confinement are different.

Recently Ogilvie[5] has developed a character expansion for Abelian and found that gauge fixing is unnecessary, i.e., Abelian projection yields string tensions of the underlying non-Abelian theory even *without* gauge fixing. Essentially the same mechanism was observed by Ambjørn and Greensite for $Z_2$ center projection of $SU(2)$ link variables[6]. See also Ref.[7]. Furthermore, by introducing a gauge fixing function $S_{gf} = \lambda \sum \mathrm{Tr} \mathrm{U}_\mu(\mathrm{x})\sigma_3 \mathrm{U}_\mu(\mathrm{x})^\dagger \sigma_3$, Ogilvie has also shown that the Abelian dominance for the string tension occurs for small $\lambda$. Hence he conjectures that Abelian dominance is gauge independent and that gauge fixing results in producing fat links for Wilson loop and is computationally advantageous for the measurements. Further Suzuki et.al. have shown that if the gauge independence of Abelian dominance is realized, the gauge independence of monopole dominance is also proved[8]. Hence to prove the gauge independence of Abelian and monopole dominances are very important especially in the intermediate region between no gauge fixing and exact MA gauge fixing.

In this letter, we analyze the gauge dependence of the Abelian projection. We now employ stochastic quantization with gauge fixing as the gauge fixing scheme which has been proposed by Zwanziger[9]

$$\frac{\partial}{\partial\tau} A_\mu^a(x,\tau) = -\frac{\delta S}{\delta A_\mu^a(x,\tau)} + \frac{1}{\alpha} D_\mu^{ab}\Delta^b + \eta_\mu^a(x,\tau), \tag{2}$$

where $x$ is Euclidean space-time and $\tau$ is fictious time. $\eta$ stands for Gaussian white noise

$$\langle \eta_\mu^a(x,\tau) \rangle = 0,$$

$$\langle \eta_\mu^a(x,\tau)\eta_\nu^b(x',\tau')\rangle = 2\delta^{ab}\delta_{\mu\nu}\delta^4(x-x')\delta(\tau-\tau').$$

Here $\Delta$ is defined as

$$\Delta(x) = \Delta^1(x)\sigma_1 + \Delta^2(x)\sigma_2,$$

$$\Delta^1(x) = -4g\sum_\mu[\partial_\mu A_\mu^1(x) - gA_\mu^2(x)A_\mu^3(x)], \tag{3}$$

$$\Delta^2(x) = -4g\sum_\mu[\partial_\mu A_\mu^2(x) + gA_\mu^1(x)A_\mu^3(x)]. \tag{4}$$

Note that $\alpha = 0$ corresponds to the MA gauge fixing and $\alpha = \infty$ is the stochastic quantization without gauge fixing.

# 2 Formulation

$SU(2)$ elements can be decomposed into diagonal and off-diagonal parts after Abelian projection,

$$U_\mu(x) = c_\mu(x)u_\mu(x), \tag{5}$$

where $c_\mu(x)$ is the off-diagonal part and $u_\mu(x)$ is the diagonal one,

$$u_\mu(x) = \begin{pmatrix} \exp(i\theta_\mu(x)) & 0 \\ 0 & \exp(-i\theta_\mu(x)) \end{pmatrix}.$$

The diagonal part can be regarded as link variable of the remaining U(1). One can construct monopole currents from field strength of U(1) links[10]:

$$\begin{aligned} \theta_{\mu\nu}(x) &= \theta_\mu(x) + \theta_\nu(x+\hat\mu) - \theta_\mu(x+\hat\nu) - \theta_\nu(x) \\ &= \bar\theta_{\mu\nu} + 2\pi n_{\mu\nu} \quad (-\pi \le \bar\theta_{\mu\nu} < \pi), \\ k_\mu(x) &= \frac{1}{4\pi}\epsilon_{\mu\nu\rho\sigma}\partial_\nu\bar\theta_{\rho\sigma}(x). \end{aligned} \tag{6}$$

Wilson loops from Abelian, monopole and photon contributions can be calculated as [11]

$$W^{\text{Abelian}} = \exp(-\frac{i}{2}\sum_{x,\mu,\nu} M_{\mu\nu}(x)\theta_{\mu\nu}(x)), \tag{7}$$

$$W^{\text{monopole}} = \exp(2\pi i \sum_{x,x',\alpha,\beta,\rho,\sigma} k_\beta(s)D(x-x')\frac{1}{2}\epsilon_{\alpha\beta\rho\sigma}\partial_\alpha M_{\rho\sigma}(x')), \tag{8}$$

$$W^{\text{photon}} = \exp(-i\sum_{x,x',\mu,\nu} \partial_\mu^-\theta_{\mu\nu}(x)D(x-x')J_\nu(x')), \tag{9}$$

$$J_\nu(x) = \partial_\mu^- M_{\mu\nu}(x),$$

where $\partial$ is a lattice forward derivative, $\partial^-$ is a backward derivative and $D(x-x')$ is the lattice Coulomb propagator. $J_\nu$ is the external source of electric charge and $M_{\mu\nu}$ has values $\pm 1$ on the surface inside of Wilson loop. In order to achieve better signals, we perform smearing for spatial link variables of non-Abelian, Abelian, monopole and photon[12] .We set $\gamma = 2.0$ for non-Abelian configurations and $\gamma = 1.0$ for the others, where $\gamma$ is the parameter which determines the mixing between a link variable itself and staples surrounding the link

$$U_\mu(x)^{\text{smeared}} = \frac{1}{N}\{\gamma\ U_\mu(x) + \sum(\text{staples})\}.$$

Here $N$ is a normalization factor.

Stochastic quantization is based on Langevin equation which describes stochastic processes in terms of fictious time[13]. A compact lattice version of this equation with gauge fixing was proposed in Ref.[14]:

$$\begin{aligned} U_\mu(x,\tau+\delta\tau) &= \omega(x,\tau)^\dagger \exp(i f_\mu^a \sigma_a) U_\mu(x,\tau)\omega(x+\hat{\mu},\tau), \qquad (10) \\ f_\mu^a &= -\frac{\partial S}{\partial A_\mu^a}\delta\tau + \eta_\mu^a(x,\tau)\sqrt{\delta\tau}, \\ \omega(x,\tau) &= \exp(i\frac{\beta}{2N_c\alpha}\Delta_{\text{lat}}^a(x,\tau)\sigma_a\delta\tau). \end{aligned}$$

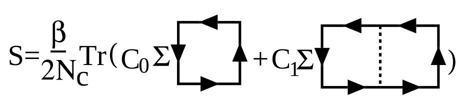
In MA gauge,

$$\begin{aligned} \Delta_{\text{lat}}(x,\tau) &= i[\sigma_3, X(x,\tau)] \\ &= 2(X_2(x,\tau)\sigma_1 - X_1(x,\tau)\sigma_2), \qquad (11) \end{aligned}$$

where

$$\begin{aligned} X(x,\tau) &= \sum_\mu (U_\mu(x,\tau)\sigma_3 U_\mu(x,\tau)^\dagger - U_\mu(x-\hat{\mu},\tau)^\dagger \sigma_3 U_\mu(x-\hat{\mu},\tau)) \qquad (12) \\ &= \sum_i X_i(x,\tau)\sigma_i. \qquad (13) \end{aligned}$$

As an improved action to reduce finite lattice spacing effects, we adopt the Iwasaki action[15]




where $C_0 + 8C_1 = 1$ and $C_1 = -0.331$. The Runge–Kutta algorithm is employed for solving the discrete Langevin equation[16]. As will be shown, the systematic error which comes from finite $\delta\tau$ is much reduced.

# 3 Numerical Results

Numerical simulations were performed on $8^3 \times 12$ and $16^3 \times 24$ lattices with $\beta = 0.995$, $\alpha = 0.1, 0.25, 0.5, 1.0$, and $\delta\tau = 0.001, 0.005, 0.01$. Measurements were done every 100–1000 Langevin time steps after 5000–50000 thermalization Langevin time steps. The numbers of Langevin time steps for the thermalization were determined by monitoring the functional $R$ and Wilson loops.

In Fig.1, we plot $(\Delta^1)^2 + (\Delta^2)^2$ as a function of the gauge parameter $\alpha$. $\alpha = 0$ corresponds to $\Delta = 0$, i.e., MA gauge. When $\alpha$ increases, the deviation from the gauge fixed plane becomes larger.

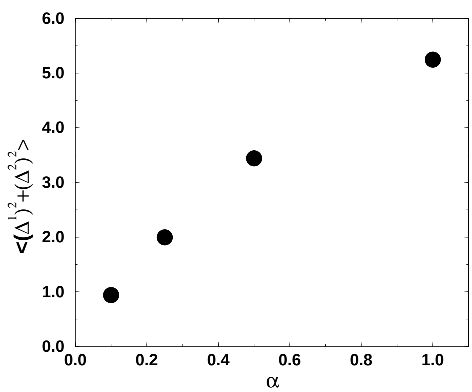



Figure 1: Gauge parameter $\alpha$ versus $(\Delta^1)^2 + (\Delta^2)^2$. Lattice size is $8^3 \times 12$, $\delta\tau = 0.005$ and $\beta = 0.995$.

We calculated the heavy quark potentials from non-Abelian, Abelian, monopole and photon contributions by

$$V(R) = -\lim_{T\to\infty} \log \frac{\langle W(R,T)\rangle}{\langle W(R,T-1)\rangle}. \tag{14}$$

We fit them with the following function,

$$V(R) = V_0 + \sigma R + \frac{e}{R}. \tag{15}$$

In order to check that our Langevin update algorithm with the stochastic gauge fixing term works correctly, we plot in Fig.2 the heavy quark potential $V(R)$, which is consistent with the result by the heatbath update. Runge–Kutta method works well, i.e., we see no difference among data with $\delta\tau = 0.001, 0.005$ and $0.01$.

In Fig.3 we show the Abelian heavy quark potentials for different $\alpha$'s together with that of non-Abelian potential. The heavy quark potentials from monopole and photon contributions are plotted in Fig.4. They can be well fitted by a linear and Coulomb terms, respectively. We see that the linear parts of potentials are essentially same from $\alpha = 0.1$ to $1.0$, and all of them show the confinement linear potential

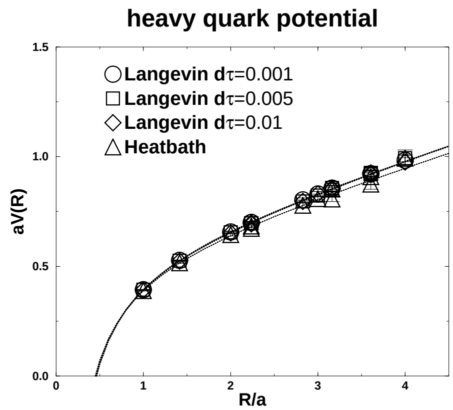



Figure 2: Non-Abelian heavy quark potentials derived from heatbath and Langevin updates for three values of $\tau$. Lattice size is $8^3 \times 12$ and $\beta = 0.995$.

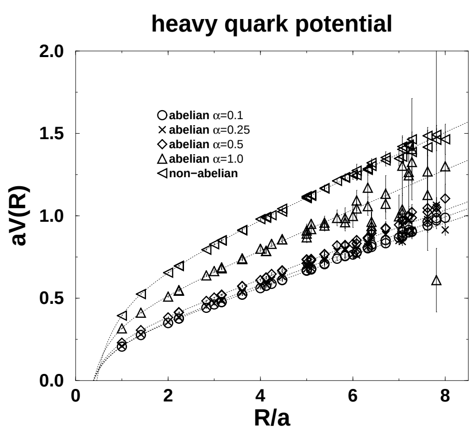



Figure 3: Heavy quark potentials from non-Abelian and Abelian contributions. Lattice size is $16^3 \times 24$, $\delta\tau = 0.005$ and $\beta = 0.995$.

behavior. Therefore even when we deviate from the MA gauge fixing condition, we can identify the monopole contribution of the heavy quark potential showing the confinement behavior. As $\alpha$ increases, statistical error becomes larger. This result suggests that the gauge fixing is favorable for decreasing numerical errors as pointed by Ogilvie[5].

In Fig.5 we plot the values of the string tensions from Abelian, monopole and photon contributions as a function of the gauge parameter $\alpha$. They are obtained by fitting the data in the range $2.0 \leq R \leq 7.0$. We have taken into account only statistical errors. The upper two lines stand for the range of the non-Abelian string tension. The Abelian and the monopole dominances are observed for all values of $\alpha$. The string tensions from the Abelian parts are about 80% of the non–Abelian one. We expect that the difference of the percentage between our result and that of Bali et.al.[12] becomes smaller when we go to larger lattice size. On the other hands, the

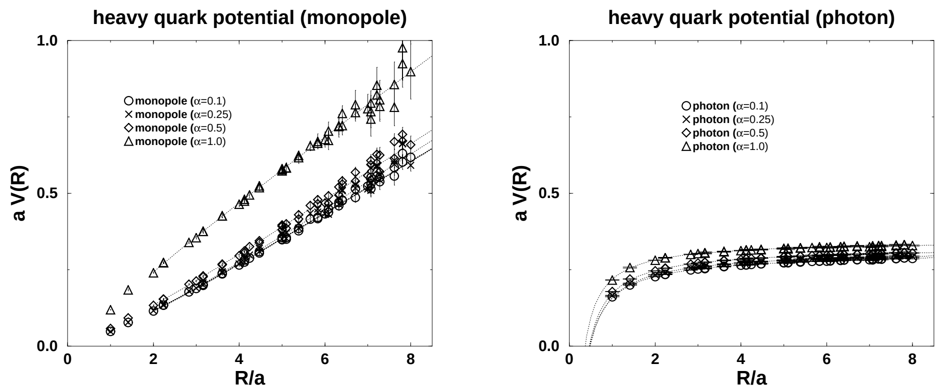



Figure 4: Heavy quark potentials from monopole(left) and photon(right) contributions. Lattice size is $16^3 \times 24$, $\delta\tau = 0.005$ and $\beta = 0.995$.

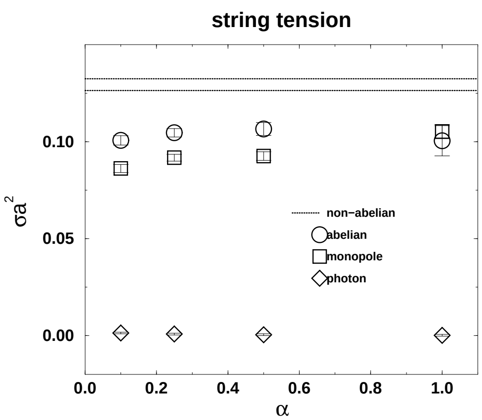



Figure 5: String tensions from non-Abelian, Abelian, monopole and photon contributions. Lattice size is $16^3 \times 24$, $\delta\tau = 0.005$ and $\beta = 0.995$.

string tension from the photon part is consistent with zero.

# 4 Concluding Remarks

We have developed a stochastic gauge fixing method which interpolates between the MA gauge and no gauge fixing. In Refs.[5],[6] and [7], effects of gauge fixing are studied for lattice algorithms where gauge fixing is done after field configurations are generated. In the stochastic gauge fixing procedure studied here, the attractive force to the gauge fixed plane along a gauge orbit is applied together with the Langevin update force. The method is tested together with the Iwasaki improved action and the Runge-Kutta algorithm. We have found it works well.

We have studied the gauge dependence of Abelian projected heavy quark potential. It is observed that the confinement force is essentially independent of the gauge

parameter. In the calculation of Abelian heavy quark potential, we have seen that as gauge parameter $\alpha$ increases, the statistical error becomes larger. This result suggests that the gauge fixing is favorable for increasing the statistics as pointed by Ogilvie[5]. It is expected that as $\alpha$ increase, Abelian string tension would approach the non–Abelian one [5, 7]. Therefore it is important to see behavior of the string tension as $\alpha$ becomes much larger than one. But data are more noisy for large $\alpha$ and we are planning to employ a noise reduction technique such as integral method[17] for obtaining statistically significant data.

It is desirable to study the gauge dependence (or independence) of other quantities, such as the monopole condensation, which may reveal the role of gauge fixing in the dual superconductor scenario. Hioki et.al. have reported the correlation between monopole density and Gribov copy in MA gauge fixing[18]. Gribov copy effect for Abelian string tension has been studied by Bali et.al.[12]. For Landau gauge, Gribov copy may be avoided by Langevin update algorithm together with the stochastic gauge fixing[14]. It is, therefore, interesting to investigate effects of Gribov copy in Abelian projection with the method.

T.S and A.N. acknowledge financial support from JSPS Grand-in Aid for Scientific Research( (B) No.10440073 and No.11695029 for T.S.) and ( (C)No.10640272 for A.N.).

# References

[1] G.'tHooft, *High Energy Physics*, ed. A.Zichichi (Editorice Compositori, Bologna, 1975).

[2] S.Mandelstam, Phys. Rep. **23C** (1976) 245.

[3] T.Suzuki and I.Yotsuyanagi, Phys.Rev. **D42** (1990) 4257;Nucl. Phys. B(Proc Suppl.) **20**

[4] T.Suzuki, COLOR CONFINEMENT AND HADRONS —Confinement 95 — , pp3–10, 1995, (1991) 236. World Scientific.

[5] M.C.Ogilvie, Phys.Rev. **D59** (1999) 074505.

[6] J.Ambjørn and J.Greensite, J. High Energy Physics **05** (1998) 004.

[7] M.Faber, J.Greensite and Š. Olejník, J. High Energy Physics. **01** (1999) 008.

[8] T.Suzuki et.al., Kanazawa Univ. Report KANAZAWA 99-18;hep-lat/9909104.

[9] D.Zwanziger, Nucl. Phys. **B192** (1981) 259.

[10] T.A.DeGrand and D.Toussaint, Phys. Rev. **D22** (1980) 2478.

[11] H.Shiba and T.Suzuki, Phys. Lett. **333B** (1994) 461.

[12] G.S.Bali, V.Bornyakov, M.Müeller–Preussker and K.Schilling, Phys. Rev. **D54** (1996) 2863.

[13] G.Parisi and Y.S.Wu, Sci. Sinica **24** (1981) 483.

[14] M.Mizutani and A.Nakamura, Vistas in Astronomy, Vol.37, pp305–308, 1993, Pergamon Press; Nucl.Phys. B(Proc.Suppl.) **34**, pp253-255, 1994.

[15] Y.Iwasaki, Nucl. Phys.**B258** (1985) 141; Univ. of Tsukuba report UTHEP–118 (1983), unpublished.

[16] G.G.Batrouni et.al., Phys. Rev. **D32** (1985) 2736; A.Ukawa,M.Fukugita, Phys. Rev. Lett. **28** (1985) 1854; M.Fukugita, Y.Oyanagi, A.Ukawa, Phys. Rev. **D36** (1987) 824.

[17] G.Parisi, R.Petronzio and F.Rapuano, Phys. Lett. **128B** (1983) 418.

[18] S.Hioki, S.Kitahara, S.Kiura, Y.Matsubara, O.Miyamura, S.Ohno and T.Suzuki, Phys. Lett. **272B** (1991) 326.